ASYMPTOTIC EXPANSION OF AN EXPONENTIAL FUNCTION OF FRACTIONAL ORDER*

(ASIMPTOTICHESKOE RAZLOZHENIE EKSPONENTSIAL'NOI FUNKTSII DROBNOGO PORIADKA)

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The exponential function of fractional order introduced by Rabotnov [1] has the form

$$\vartheta_\alpha(\beta,\ t) = t^\alpha \sum_{n=0}^{n=\infty} \frac{\beta^n t^{n(1+\alpha)}}{\Gamma[(n+1)(1+\alpha)]} \qquad (1+\alpha > 0) \tag{1}$$

We are interested in the asymptotic expansion for large t of this function and its derivatives with respect to t and β, and also of the integrals

$$\int_0^t \vartheta_\alpha(\beta,\ t)\,dt, \qquad \int_0^t \frac{\partial \vartheta_\alpha(\beta,\ t)}{\partial\beta}\,dt$$

The following theorem, which was established in [2], is basic for this article.

Theorem 1. Let us consider the function

$$E_\gamma(z,\ q) = \sum_{n=0}^{n=\infty} \frac{h(n)}{\Gamma(\gamma n + q)} z^n \qquad (z = x + iy,\ \gamma > 0) \tag{2}$$

where q is an arbitrary constant, real or complex.

Let $h(n)$ be such that if one considers the function $h(w)$, where $w = x + iy$ in any half-space $x > x_0$, then

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* While reading the proof sheets, the author became aware of the works of M.M. Dzhrbashian [4,5] in which were investigated, in particular, important properties of functions of the type $\vartheta_\alpha(\beta,\ t)$.

$$\frac{h(w)}{\Gamma(\gamma w + q)}$$

is a single-valued analytic function w which for large values of the modulus can be represented in the form

$$h(w) = C_0 + \frac{C_1}{\gamma w + q} - \ldots - \frac{C_s + \delta(\gamma w, s)}{(\gamma w + q)(\gamma w + q + 1) \ldots (\gamma w + q + s)} \tag{3}$$

where $c_0, c_1, \ldots$ do not depend on w, and where the function $\delta(\gamma w, s)$ is such that

$$\lim \delta(\gamma w, s) = 0 \quad \text{for } |w| \to \infty$$

Thus, for the function $E_\gamma(z, q)$ with large $|z|$, there exist asymptotic representations:

when $0 < \gamma < 2, \qquad \frac{1}{2}\pi < \arg z < \left(2 - \frac{1}{2}\gamma\right)\pi$

$$E_\gamma(z, q) \sim -\sum_{n=1}^{n=\infty} \frac{h(-n)}{\Gamma(q - \gamma n)} z^{-n} \tag{4}$$

when $0 < \gamma < 2, \ |\arg z| < \frac{1}{2}\pi\gamma$

$$E_\gamma(z, q) \sim \frac{1}{\gamma} z^{\frac{1-q}{\gamma}} \exp\left(z^{\frac{1}{\gamma}}\right) \sum_{n=0}^{n=\infty} C_n z^{-\frac{n}{\gamma}} \quad (c_0, c_1, \ldots \text{same as in } (3)) \tag{5}$$

when $0 < \gamma < 2, \ |\arg z| = \frac{1}{2}\pi\gamma$

$$E_\gamma(z, q) \sim \frac{1}{\gamma} z^{\frac{1-q}{\gamma}} \exp\left(z^{\frac{1}{\gamma}}\right) \sum_{n=0}^{n=\infty} C_n z^{-\frac{n}{\gamma}} - \sum_{n=1}^{n=\infty} \frac{h(-n)}{\Gamma(q - \gamma n)} z^{-n} \tag{6}$$

when $\gamma \geqslant 2, \ |\arg z| < \pi$

$$E_\gamma(z, q) \sim \frac{1}{\gamma} \sum_\mu \left\{ \exp Z_\mu) Z_\mu^{1-q} \sum_{n=0}^{n=\infty} C_n (Z_\mu)^{-n} \right\} \qquad \left(Z_\mu = z^{\frac{1}{\gamma}} \exp \frac{2\pi i \mu}{\gamma}\right) \tag{7}$$

The first summation is performed for all μ for which

$$|\arg z + 2\pi\mu| \leqslant \frac{1}{2}\pi\gamma$$

Let us restrict ourselves to the case $0 < \alpha + 1 < 2$. If $\beta > 0$, then it follows from (5), and if $\beta < 0$ from (4), that

$$\begin{aligned} \vartheta_\alpha(\beta, t) &\sim \frac{1}{1+\alpha} \beta^{-\frac{\alpha}{1+\alpha}} \exp\left(t\beta^{\frac{1}{1+\alpha}}\right) \qquad (\beta > 0) \\ \vartheta_\alpha(\beta, t) &\sim -t^\alpha \sum_{n=2}^{n=\infty} \frac{(\beta t^{1+\alpha})^{-n}}{\Gamma[(\alpha+1) - (\alpha+1)n]} \qquad (\beta < 0) \end{aligned} \tag{8}$$

If $\beta < 0$ we deduce from (8) that for large t

$$\int_t^{\infty} \vartheta_\alpha(\beta, t)\, dt \sim -t^{\alpha+1} \sum_{n=2}^{n=\infty}{}' \frac{(\beta t^{1+\alpha})^{-n}}{\Gamma[\alpha+2-(\alpha+1)n]} \qquad (9)$$

Analogously, using Theorem 1, one can show that the asymptotic expansions of $\partial \vartheta_\alpha(\beta, t)/\partial t$ and $\partial \vartheta_\alpha(\beta, t)/\partial \beta$ will be equal to the derivatives with respect to t and β of the asymptotic expansion (8) for $\vartheta_\alpha(\beta, t)$.

Furthermore, with the aid of the same theorem one can find

$$\int_0^t \vartheta_\alpha(\beta, t)\, dt \sim \frac{1}{\beta(1+\alpha)} \exp(t\beta^{\frac{1}{1+\alpha}}) \qquad (\beta > 0)$$

$$\int_0^t \vartheta_\alpha(\beta, t)\, dt \sim -\frac{1}{\beta} - t^{\alpha+1} \sum_{n=2}^{n=\infty} \frac{(\beta t^{1+\alpha})^{-n}}{\Gamma[\alpha+2-(\alpha+1)n]} \qquad (\beta < 0) \qquad (10)$$

$$\int_0^t \frac{\partial \vartheta_\alpha(\beta, t)}{\partial \beta}\, dt \sim \frac{t}{(1+\alpha)^2} \beta^{-\frac{(1+2\alpha)}{1+\alpha}} \exp(t\beta^{\frac{1}{1+\alpha}}) \left[1 - \frac{1+\alpha}{t} \beta^{-\frac{1}{1+\alpha}}\right] \qquad (\beta > 0)$$

$$\int_0^t \frac{\partial \vartheta_\alpha(\beta, t)}{\partial \beta}\, dt \sim \frac{t^{\alpha+1}}{\beta} \sum_{n=2}^{n=\infty} \frac{n(\beta t^{1+\alpha})^{-n}}{\Gamma[\alpha+2-(\alpha+1)n]} \qquad \beta < 0 \qquad (11)$$

The asymptotic expansion (10) with $\beta < 0$ was found earlier by Rozovskii [3].

We note that if β is a complex number, then

$$\lim_{t\to\infty} \vartheta_\alpha(\beta, t) = 0 \quad \text{for} \quad \tfrac{1}{2}\pi(1+\alpha) \leqslant \arg\beta \leqslant \left(2 - \frac{\alpha+1}{2}\right)\pi$$

$$\lim_{t\to\infty} \vartheta_\alpha(\beta, t) = \infty \quad \text{for} \quad |\arg\beta| < \tfrac{1}{2}\pi(\alpha+1) \quad .$$

If $\alpha + 1 \geqslant 2$, then one must use Formula (7).

BIBLIOGRAPHY

1. Rabotnov, Iu.N., Ravnovesie uprugoi sredy s posledeistviem (Equilibrium of an elastic medium with after-effect). *PMM* Vol. 12, No. 1, 1948.

2. Fry, C.G. and Hughes, H.K., Asymptotic developments of certain integral functions. *Duke Mathem. J.* Vol. 9, pp. 791-802, 1942.

3. Rozovskii, M.I., Nelineinye integral'no-operatornye uravneniia polzuchesti i zadacha o kruchenii tsilindra pri bol'shikh uglakh krutki (Nonlinear integral operator equations of creep and the problem on the torsion of a cylinder with large torsion angles). *Izv. Akad. Nauk SSSR, OTN, Mekhanika i mashinostroenie* No. 5, 1959.

4. Dzhrbashian, M.M., Ob integral'nom predstavlenii funktsii, nepreryvnykh na neskol'kikh luchakh (obobshchenie integrala Fur'e) (On the integral representation of functions continuous on several rays (a generalization of the Fourier integral)). *Izv. Akad. Nauk SSSR, seriia matem.* 18, 1954.

5. Dzhrbashian, M.M., Ob odnom novom integral'nom preobrazovanii i ego primenenii v teoriiakh tselykh funktsii (On a new integral transformation and its application in the theories of complete functions). *Izv. Akad. Nauk SSSR, seriia matem.* 19, 1955.

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